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Derivation of Slip Boundary Conditions for the Navier–Stokes System from the Boltzmann Equation

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Rarefied gas flow behavior is usually described by the Boltzmann equation, the Navier–Stokes system being valid when the gas is less rarefied. Slip boundary conditions for the Navier–Stokes equations are derived in a rigorous and systematic way from the boundary condition at the kinetic level (Boltzmann equation). These slip conditions are explicitly written in terms of asymptotic behavior of some linear half-space problems. The validity of this analysis is established in the simple case of the Couette flow, for which it is proved that the right boundary conditions are obtained.

KEY WORDS: Boltzmann equation; Knudsen layer; Navier–Stokes equations; slip boundary conditions.

1. INTRODUCTION AND MAIN RESULTS

Studies of such phenomena as flow past aircraft flying at high altitudes are relevant to rarefied gas dynamics. Investigation in this domain is usually done using a distribution function $F(x, \xi)$, density of particles at point x with velocity ξ, satisfying the Boltzmann equation.[7,9,12,18,29]

A physically important parameter of the flow is the Knudsen number, which is the ratio of the mean free path by a characteristic length. In the limiting case of small Knudsen number, it is well known that problems may be solved within the fluid dynamic theory in which the unknowns are ρ, u, T (the density, velocity, and temperature of the flow) satisfying the Navier–Stokes system. The link between the kinetic description (Boltzmann equation) and this continuum limit is made by the Chapman–Enskog expansion. This formal expansion is proved to be valid when the fluid fills the whole space and under some technical assumptions.[1,19] If there are

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obstacles in the flow, some conditions on ρ, u, T are to be fixed at the boundary. When the Knudsen number is very small, the assumption that the fluid sticks to the bodies gives physically good boundary conditions: the velocity and the temperature near a wall are those of the wall. However, when the Knudsen number is not so small, one has to take into account slip boundary conditions. There has been much work on this topic and on the way to obtain numerically the best boundary conditions.[16,20,23,24] The reason for the slip phenomenon is easy to explain: the Chapman–Enskog expansion generally does not satisfy the kinetic boundary condition and so is not valid in a region near the body, called the Knudsen layer, the thickness of which is of order of the mean free path.

The aim of this paper is to obtain rigorously and in a systematic way the slip boundary conditions. These slip boundary conditions for the Navier–Stokes system are derived from the kinetic boundary condition (for example, the well-known Maxwell accommodation condition or more general ones; see ref. 7). The analysis is based on the study of the Knudsen layer, which is related to half-space problem.

In the following sections, we show how to obtain the boundary conditions for the Navier–Stokes system by a linear half-space problem analysis. We then prove that in the particular case of the Couette flow (see Section 2) there exists a solution (ρ, u, T) of the Navier–Stokes system satisfying the obtained boundary conditions. Then, adding to the Chapman–Enskog expansion a Knudsen layer term χ, we obtain a solution which satisfies at a certain order with respect to the Knudsen number the Boltzmann equation and the kinetic boundary condition [see properties (2.5.1–2.5.2)]. Let us emphasize that we do not prove here that there exists a solution to the Boltzmann problem. Such a result might be obtained by the method of Caflisch.[5] We do not compute numerically the coefficients which appear in the boundary conditions but we prove that they are related to the asymptotic behavior of half-space problems. An approximate variational method to compute these coefficients was proposed by Golse[14] and by Loyalka.[24]

The following sections are organized as follows:

In Section 2, we study the simple case of Couette flow, first with complete accommodation and then with the Maxwell accommodation boundary. This example is used to explain the method, to obtain simpler expressions, and to give explicit estimations. Properties (2.5.1–2.5.2) prove that the right boundary conditions are obtained by only a linear half-space problem analysis for the Knudsen layer. As in ref. 20, the Couette flow, though simpler, is carefully studied because it contains the most important effects of slip flows and derivation of slip boundary conditions.

In Section 3, we analyze the general case of the multidimensional problem. It is shown how the one-dimensional analysis is relevant to studying the Knudsen layer and to obtaining the slip boundary conditions, at least from a formal point of view.

The last section is devoted to some remarks concerning extensions of this method and its possible limitations.

2. STATIONARY COUETTE FLOW

2.1. Introduction

We consider a stationary Couette flow between two plates positioned at $x=0$ and at $x=L$ and orthogonal to the x axis (see Fig. 1). We assume complete accommodation at the boundary. We obtain the following equations for the distribution of particles $F_\varepsilon(x, \xi)$ at point x and with velocity ξ:

$$\xi_1 \partial_x F_\varepsilon - \frac{1}{\varepsilon} Q(F_\varepsilon, F_\varepsilon) = 0, \qquad 0 \leqslant x \leqslant L, \qquad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 \quad (2.1)$$

$$\exists \alpha_1 \in \mathbf{R} / F_\varepsilon(0, \xi) = \alpha_1 M_{u_1, T_1}(\xi), \qquad \xi_1 > 0 \quad (2.2)$$

$$\exists \alpha_2 \in \mathbf{R} / F_\varepsilon(L, \xi) = \alpha_2 M_{u_2, T_2}(\xi), \qquad \xi_1 < 0 \quad (2.3)$$

$$\int \xi_1 F_\varepsilon(x, \xi)\, d\xi = 0 \quad (2.4)$$

$$\int_0^L \int F_\varepsilon(x, \xi)\, d\xi\, dx = N \quad (2.5)$$

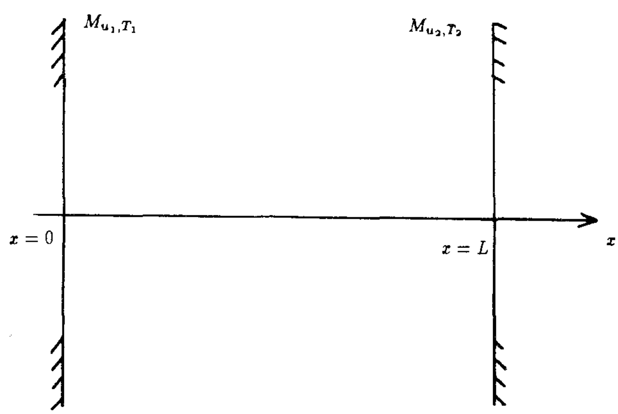

Fig. 1. Couette flow.

Equation (2.1) is the one-dimensional stationary Boltzmann equation. The parameter ε is the Knudsen number and $Q(F_\varepsilon, F_\varepsilon)$ is the collision term.[7,9,12,18,29] We restrict ourselves to collision kernels for a hard-sphere gas satisfying the angular cutoff assumption as proposed by Grad (see ref. 10). We denote by M_{u_1,T_1} the Maxwellian with temperature T_1 and mean velocity u_1, which are, respectively, the temperature and the velocity of the left plane (at $x=0$). We assume that u_1 is normal to the x axis. Similarly, we denote by M_{u_2,T_2} the Maxwellian corresponding to the right plate (at $x=L$)

$$M_{u_i,T_i}(\xi)=\frac{1}{(2\pi RT_i)^{3/2}}\exp\left(-\frac{|\xi-u_i|^2}{2RT_i}\right)$$
$$u_i=(u_{i1},u_{i2},u_{i3}) \qquad \text{with} \quad u_{i1}=0, \quad i=1,2 \tag{2.6}$$

Using Eq. (2.1) and the usual property of conservation for the collision term, we obtain

$$\partial_x\left(\int \xi_1 F_\varepsilon(x,\xi)\,d\xi\right)=0 \tag{2.7}$$

Then (2.2)–(2.4) are equivalent to a complete accommodation condition at the boundary.

From (2.5), the total number of particles is equal to N. This condition is added to (2.1)–(2.4) in order to avoid the trivial solution $F_\varepsilon=0$. For this purpose, it is purpose, it is also possible to consider (2.1)–(2.4) with a given coefficient α_1 in (2.2).

2.2. The Chapman–Enskog Expansion

The link between the Boltzmann equation and the Navier–Stokes system is realized by the Chapman–Enskog expansion[1,9]

$$F_{CE}(x,\xi)=M_{u_\varepsilon,T_\varepsilon}(\xi)[\rho_\varepsilon-\varepsilon h_\varepsilon(x,\xi)] \tag{2.8}$$

where ρ_ε, u_ε, T_ε depend on x and

$$M_{u_\varepsilon,T_\varepsilon}(\xi)=\frac{1}{(2\pi RT_\varepsilon)^{3/2}}\exp\left(-\frac{|\xi-u_\varepsilon|^2}{2RT_\varepsilon}\right) \tag{2.9}$$

and $h_\varepsilon(x,\xi)$ is expressed in terms of Sonine polynomials $A_1(\xi)$ and $B_i(\xi)$ by

$$h_\varepsilon(x,\xi)=\sum_{i=1}^{i=3} b(T_\varepsilon,|\xi|)\,B_i(\xi)\,\partial_x u_{\varepsilon i}+a(T_\varepsilon,|\xi|)\,A_1(\xi)(RT_\varepsilon)^{1/2}\,\partial_x\log(T_\varepsilon) \tag{2.10}$$

where

$$\bar{\xi} = \frac{\xi - u_\varepsilon}{(RT_\varepsilon)^{1/2}} \tag{2.11}$$

$$A_1(\bar{\xi}) = \bar{\xi}_1 \frac{|\bar{\xi}|^2 - 5}{2} \tag{2.12}$$

$$B_i(\bar{\xi}) = \bar{\xi}_1 \bar{\xi}_i - \frac{|\bar{\xi}|^2}{3} \delta_{1,i} \tag{2.13}$$

$a(T_\varepsilon, |\bar{\xi}|)$ is given by

$$-2Q(M_{u_\varepsilon, T_\varepsilon}, M_{u_\varepsilon, T_\varepsilon} a(T_\varepsilon, |\bar{\xi}|)\, A_1(\bar{\xi})) = A_1(\bar{\xi}) M_{u_\varepsilon, T_\varepsilon} \tag{2.14}$$

and $b(T_\varepsilon, |\bar{\xi}|)$ is given by

$$-2Q(M_{u_\varepsilon, T_\varepsilon}, M_{u_\varepsilon, T_\varepsilon} b(T_\varepsilon, |\bar{\xi}|)\, B_i(\bar{\xi})) = B_i(\bar{\xi}) M_{u_\varepsilon, T_\varepsilon} \tag{2.15}$$

The one-dimensional stationary Navier–Stokes system for $\rho_\varepsilon, u_\varepsilon, T_\varepsilon$ is

$$\partial_x(\rho_\varepsilon u_{\varepsilon 1}) = 0 \tag{2.16}$$

$$\partial_x(\rho_\varepsilon u_{\varepsilon 1}^2) + \partial_x(\rho_\varepsilon R T_\varepsilon) = \varepsilon \partial_x(\tfrac{4}{3}\mu(T_\varepsilon)\, \partial_x u_{\varepsilon 1}) \tag{2.17}$$

$$\partial_x(\rho_\varepsilon u_{\varepsilon 1} u_{\varepsilon i}) = \varepsilon \partial_x(\mu(T_\varepsilon)\, \partial_x u_{\varepsilon i}), \qquad i = 2, 3 \tag{2.18}$$

$$\partial_x(\rho_\varepsilon u_{\varepsilon 1}(\tfrac{5}{2} R T_\varepsilon + \tfrac{1}{2} u_\varepsilon^2)) = \varepsilon \partial_x(\tfrac{4}{3}\mu(T_\varepsilon)\, u_{\varepsilon 1} \partial_x u_{\varepsilon 1} + \lambda(T_\varepsilon)\, \partial_x T_\varepsilon) \tag{2.19}$$

where the viscosity $\mu(T_\varepsilon)$ and thermal conductivity $\lambda(T_\varepsilon)$ are expressed in terms of $a(T_\varepsilon, \bar{\xi})$ and $b(T_\varepsilon, \bar{\xi})$ by

$$\lambda(T_\varepsilon) = \frac{R^2 T_\varepsilon}{6(2\pi)^{1/2}} \int_0^\infty (r^8 - 5r^6)\, a(T_\varepsilon, r) \exp\left(-\frac{r^2}{2}\right) dr \tag{2.20}$$

$$\mu(T_\varepsilon) = \frac{2RT_\varepsilon}{15(2\pi)^{1/2}} \int_0^\infty r^6 b(T_\varepsilon, r) \exp\left(-\frac{r^2}{2}\right) dr \tag{2.21}$$

We recall the result of ref. 1: let $\rho_\varepsilon, u_\varepsilon, T_\varepsilon$ satisfy the Navier–Stokes system (2.16)–(2.19); then there exists a function $W(x, \xi)$ such that

$$\forall x \in [0, L], \qquad \int \psi(\xi)\, W(x, \xi)\, d\xi = 0 \qquad \text{for} \quad \psi(\xi) = 1, \xi_1, \xi_2, \xi_3, |\xi|^2 \tag{2.22}$$

and

$$-2Q(\rho_\varepsilon M_{u_\varepsilon, T_\varepsilon}, W) = S_\varepsilon \tag{2.23}$$

with

$$S_\varepsilon(x,\xi)=\xi_1\,\partial_x(M_{u_\varepsilon,T_\varepsilon}h_\varepsilon)+Q(M_{u_\varepsilon,T_\varepsilon}h_\varepsilon,\,M_{u_\varepsilon,T_\varepsilon}h_\varepsilon)$$
$$-\left\{\frac{\xi_1-u_{\varepsilon 1}}{RT_\varepsilon}\,\partial_x\left(\frac{4}{3}\,\mu(T_\varepsilon)\,\partial_x u_{\varepsilon 1}\right)\right.$$
$$\left.+\sum_{i=2,3}\frac{\xi_i-u_{\varepsilon i}}{RT_\varepsilon}\,\partial_x(\mu(T_\varepsilon)\,\partial_x u_{\varepsilon i})\right\}M_{u_\varepsilon,T_\varepsilon}(x,\xi)$$
$$-\frac{|\xi-u_\varepsilon|-3RT_\varepsilon}{3R^2T_\varepsilon^2}\,\partial_x\left(\mu(T_\varepsilon)\left\{\frac{4}{3}\,(\partial_x u_{\varepsilon 1})^2+\frac{1}{2}\,(\partial_x u_{\varepsilon 2})^2\right.\right.$$
$$\left.\left.+\frac{1}{2}\,(\partial_x u_{\varepsilon 3})^2\right\}+\lambda(T_\varepsilon)\,\partial_x T_\varepsilon\right)M_{u_\varepsilon,T_\varepsilon}(x,\xi)$$

and so we obtain

$$\xi_1\,\partial_x(F_{CE}+\varepsilon^2W)-\frac{1}{\varepsilon}\,Q(F_{\rm CE}+\varepsilon^2W,\,F_{\rm CE}+\varepsilon^2W)$$
$$=\varepsilon^2(\xi_1\,\partial_x W+2Q(M_{u_\varepsilon,T_\varepsilon}h_\varepsilon,\,W)-\varepsilon Q(W,W)) \qquad (2.24)$$

Usually, the functions $a(T,|\xi|)$, $b(T,|\xi|)$ are expressed in terms of Sonine polynomials.[9,18] For simplicity, we keep only the first term of such an expansion (that is, we assume that these functions do not depend on $|\xi|$, this being rigorously the case for Maxwellian molecules). From a practical point of view, such an approximation gives very good results.[4] In this case, (2.20) and (2.21) give

$$\lambda(T_\varepsilon)=\tfrac{5}{2}R^2T_\varepsilon a(T_\varepsilon), \qquad \mu(T_\varepsilon)=RT_\varepsilon b(T_\varepsilon)$$

and (2.10) becomes

$$h_\varepsilon(x,\xi)=\sum_{i=1}^{i=3}\frac{\mu(T_\varepsilon)}{(RT_\varepsilon)^2}\left((\xi_1-u_{\varepsilon 1})(\xi_i-u_{\varepsilon i})-\frac{|\xi-u_\varepsilon|^2}{3}\,\delta_{1,i}\right)\partial_x u_{\varepsilon i}$$
$$+\frac{2\lambda(T_\varepsilon)}{5(RT_\varepsilon)^2}\,(\xi_1-u_{\varepsilon 1})\,\frac{|\xi-u_\varepsilon|^2-5RT_\varepsilon}{2RT_\varepsilon}\,\partial_x T_\varepsilon \qquad (2.25)$$

In the following sections, we consider the case of inverse-power-law interaction between molecules. In this case and under the above approximation, μ and λ satisfy the following law[4,18]:

$$\mu(T_\varepsilon)=C_\mu(T_\varepsilon)^\alpha \qquad (2.26)$$

$$\lambda(T_\varepsilon)=C_\lambda(T_\varepsilon)^\alpha \qquad (2.27)$$

where C_λ and C_μ are constants. In particular, the model for the collision cross section is the hard sphere model, $\alpha = 1/2$.

Remark. The assumptions that $a(T, |\xi|)$ and $b(T, |\xi|)$ do not depend on $|\xi|$ and therefore that the relations (2.26), (2.27) are satisfied are only made to get simpler expressions for the Chapman–Enskog expansion and the Navier–Stokes system. The results obtained in this paper can be generalized for more complex functions $a(T, |\xi|)$, $b(T, |\xi|)$.

2.3. Derivation of the Slip Boundary Conditions for the Navier–Stokes Equations

Now, we are going to derive the boundary conditions for Navier–Stokes. Indeed, the Chapman–Enskog expansion usually does not satisfy the boundary conditions (2.2)–(2.4). To take into account these boundary conditions, we add to $F_{\mathrm{CE}}(x, \xi)$ two kinetic boundary layer terms at $x = 0$ and at $x = L$:

$$\chi_1\left(\frac{x}{\varepsilon}, \xi\right) \quad \text{and} \quad \chi_2\left(\frac{L-x}{\varepsilon}, \xi\right)$$

(the scaling in x/ε means that these two corrective terms are to be concentrated at the boundary). When the Knudsen number tends to zero, we expect the velocity and the temperature jump at the wall to vanish (the proof of this result is given in Section 2.5). At the zeroth order in ε, $\rho_\varepsilon M_{u_\varepsilon, T_\varepsilon}$ is equal to $\rho_\varepsilon(0) M_{u_1, T_1}$ at $x = 0$. At the first order in ε

$$M_{u_\varepsilon(0), T_\varepsilon(0)} \simeq M_{u_1, T_1}\left(1 + \sum_{i=1}^{i=3} \frac{u_{\varepsilon i}(0) - u_{1i}}{RT_1}(\xi_i - u_{1i}) + \frac{T_\varepsilon(0) - T_1}{T_1} \frac{|\xi - u_1|^2 - 3RT_1}{2RT_1}\right) \tag{2.28}$$

Similarly we shall prove that the gradient of the fluid quantities $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ remains bounded as ε tends to zero. We thus have, at the first order in ε,

$$\varepsilon M_{u_\varepsilon(0), T_\varepsilon(0)} h_\varepsilon(0, \cdot) \simeq = \mathrm{M}_{\mathrm{u}_1, T_1} h_\varepsilon(0, \cdot) \tag{2.29}$$

From problem (2.1)–(2.4) we obtain

$$\xi_1 \partial_x \chi_1 - 2Q(M_{u_1, T_1}, \chi_1) = 0, \qquad 0 \leqslant x < \infty \tag{2.30}$$

$$\begin{aligned} \chi_1(0, \xi) = {} & [\alpha_1 - \rho_\varepsilon(0)]\, M_{u_1, T_1}(\xi) - \rho_\varepsilon(0)\, M_{u_1, T_1}\left(\sum_{i=1}^{i=3} \frac{u_{\varepsilon i}(0) - u_{1i}}{RT_1}(\xi_i - u_{1i})\right. \\ & \left. + \frac{T_\varepsilon(0) - T_1}{T_1} \frac{|\xi - u_1|^2 - 3RT_1}{2RT_1}\right) \\ & + \varepsilon M_{u_1, T_1} h_\varepsilon(0, \xi), \qquad \xi_1 > 0 \end{aligned} \tag{2.31}$$

This linearized problem has been studied by Bardos *et al.*[2] and Cercignani.[8] They proved existence and uniqueness of a bounded solution with zero mass flux

$$\int \xi_1 \chi_1(x, \xi)\, d\xi = 0 \tag{2.32}$$

When x tends to infinity, this solution converges exponentially fast to

$$M_{u_1, T_1}(\xi)\left(a + b_2 \frac{\xi_2 - u_{12}}{(RT_1)^{1/2}} + b_3 \frac{\xi_3 - u_{13}}{(RT_1)^{1/2}} + c \frac{|\xi - u_1|^2 - 3RT_1}{2RT_1}\right) \tag{2.33}$$

where the coefficients a, b_2, b_3, c depend on $\rho_\varepsilon(0)$, $u_{\varepsilon i}(0)$, $T_\varepsilon(0)$, $\partial_x u_{\varepsilon i}(0)$, $\partial_x T_\varepsilon(0)$. The dependence with respect to u_1, T_1 is obtained by changing ξ by ξ and x by $(RT_1)^{1/2}x$, so that M_{u_1, T_1} is transformed in the absolute Maxwellian M_0 (whose mean velocity is equal to zero and whose temperature is equal to $1/R$) and the asymptotic limit on χ_1 is unchanged (we have used the fact that $u_{11} = 0$). We then compute the asymptotic limit of the solution of the problem

$$\xi_1 \partial_x \chi - 2Q(M_0, \chi) = 0 \tag{2.34}$$

$$\chi(0, \xi) = \phi(\xi), \qquad \xi_1 > 0 \tag{2.35}$$

$$\int \xi_1 \chi(x, \xi)\, d\xi = 0 \tag{2.36}$$

for various function ϕ.

We notice that if ϕ is odd (respectively even) with respect to ξ_2, then the limit is also odd (respectively even). The fact that changing ξ_2 by ξ_3 in ϕ, and vice versa produces the same change in the asymptotic limit also reduces the number of asymptotic limits to compute for the above linear half-space problem. Moreover, if ϕ is equal to M_0, $\xi_2 M_0$, $\xi_3 M_0$, or $\frac{1}{2}(|\xi|^2 - 3)M_0$, then the solution χ is constant and equal to ϕ.

Since χ_1 is a boundary layer term, χ_1 tends to zero at infinity. This last condition is equivalent to some conditions on the unknown of the Navier–Stokes system at the boundary; for example, the mass flux of χ_1 must be zero. According to (2.4), F_ε also has zero mass flux. Thus, the mass flux of F_{CE} is zero and from (2.8), (2.10) we obtain

$$u_{\varepsilon 1}(0) = 0 \tag{2.37}$$

The fact that χ_1 is a boundary layer term ensures the following boundary conditions:

$$\rho_\varepsilon(0)[u_{\varepsilon i}(0) - u_{1i}] = \varepsilon C_1 \partial_x u_{\varepsilon i}(0), \qquad i = 2, 3 \tag{2.38}$$

$$\rho_\varepsilon(0)[T_\varepsilon(0) - T_1] = \varepsilon C_2 \partial_x u_{\varepsilon 1}(0) + \varepsilon C_3 \partial_x T_\varepsilon(0) \tag{2.39}$$

(2.38) and (2.39) are, respectively, equivalent to $b_i = 0$ for $i = 2, 3$ and $c = 0$ in the limit at infinity of χ_1 given by (2.33). Moreover, the fact that the asymptotic limit of the solution χ_1 of (2.30), (2.31) with no mass flux has no component on M_{u_1, T_1} gives another relation which determines the value of α_1,

$$\alpha_1 = \rho_\varepsilon(0) + \varepsilon C_4 \partial_x u_{\varepsilon 1}(0) + \varepsilon C_5 \partial_x T_\varepsilon(0) \tag{2.40}$$

$C_1, C_2,\dots, C_5$ are constants depending on T_1 and are given by

$$C_1 = \frac{\mu(T_1)}{(RT_1)^{1/2}} c_1, \qquad C_2 = \frac{\mu(T_1)}{R} c_2, \qquad C_3 = \frac{2\lambda(T_1)}{5R(RT_1)^{1/2}} c_3 \tag{2.41}$$

$$C_4 = \frac{\mu(T_1)}{R} c_4, \qquad C_5 = \frac{2\lambda(T_1)}{5R(RT_1)^{1/2}} c_5 \tag{2.42}$$

$c_1, c_2,\dots, c_5$ are constants independent of T_1 and are given by the asymptotic limit of the solutions of (2.34)–(2.36) for the following functions $\phi(\xi)$:

$$\text{for } \phi(\xi) = (\xi_1 \xi_2) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = c_1 \xi_2 M_0(\xi) \tag{2.43}$$

$$\text{for } \phi(\xi) = \left(\xi_1^2 - \frac{|\xi|^2}{3}\right) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = \left(c_4 + c_2 \frac{|\xi|^2 - 3}{2}\right) M_0(\xi) \tag{2.44}$$

$$\text{for } \phi(\xi) = \xi_1 \left(\frac{|\xi|^2 - 5}{2}\right) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = \left(c_5 + c_3 \frac{|\xi|^2 - 3}{2}\right) M_0(\xi) \tag{2.45}$$

We have thus obtained four scalar conditions ar $x = 0$, (2.37)–(2.39).

Similar equations are derived at $x = L$.

Equations (2.37)–(2.39) are the slip boundary conditions obtained by our analysis. Their expressions are similar to those written in refs. 16 and 20. This section has established the link between the coefficients which appear in these slip boundary conditions and asymptotic limits of some linear half-space problems.

2.4. Solution of the Navier–Stokes System

Property 2.4.1. For ε small enough, there exists a solution ρ_ε, u_ε, T_ε of the Navier–Stokes equations (2.16)–(2.19) with the boundary conditions (2.37)–(2.39) and such that the integral of the density ρ_ε over

[0, L] is equal to N. Moreover, the derivatives (at any order) of the fluid quantities remain uniformly bounded when ε tends to zero (see Figs. 2 and 3) and there exist constant Cst such that

$$|T_\varepsilon(0)-T_1| \leqslant Cst\cdot\varepsilon, \qquad |u_{\varepsilon i}(0)-u_{1i}| \leqslant Cst\cdot\varepsilon, \qquad i=2,3 \tag{2.46}$$

$$|T_\varepsilon(L)-T_2| \leqslant Cst\cdot\varepsilon, \qquad |u_{\varepsilon i}(L)| \leqslant Cst\cdot\varepsilon, \qquad i=2,3 \tag{2.47}$$

Proof. From (2.16) and (2.37) we have

$$\rho_\varepsilon(x)\,u_{\varepsilon 1}(x)=0, \qquad \forall x\in[0, L] \tag{2.48}$$

Thus, the Navier–Stokes system becomes

$$u_{\varepsilon 1}=0 \tag{2.49}$$

$$\rho_\varepsilon RT_\varepsilon = c_\varepsilon \tag{2.50}$$

$$\partial_x(\mu(T_\varepsilon)\,\partial_x u_{\varepsilon i})=0, \qquad i=2,3 \tag{2.51}$$

$$\partial_x(\lambda(T_\varepsilon)\,\partial_x T_\varepsilon)=0 \tag{2.52}$$

In (2.50), c_ε is a constant related to N.

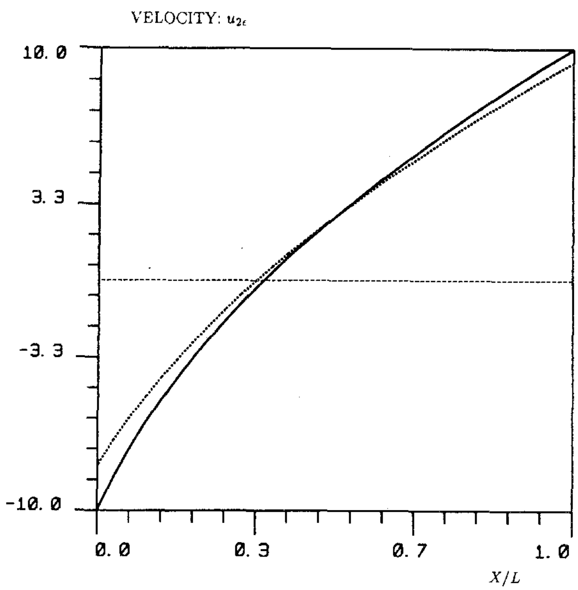

Fig. 2. Couette flow. Velocity: $u_{12}=-10$, $u_{22}=10$. (—) $\varepsilon=0$, (···) $\varepsilon\neq 0$.

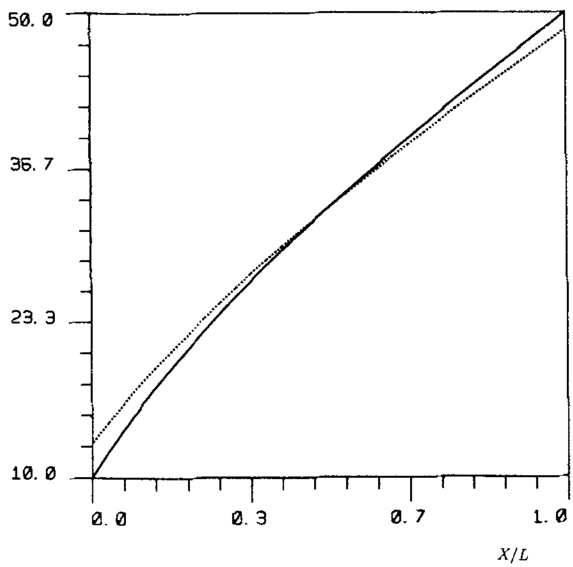

Fig. 3. Couette flow. Temperature: $T_1 = 10$, $T_2 = 50$. (——) $\varepsilon = 0$, $(\cdots)$ $\varepsilon \neq 0$.

The boundary conditions (2.38)–(2.39) become, at $x = 0$,

$$u_{\varepsilon i}(0) - u_{1i} = \varepsilon \mu(T_1) \frac{(RT_1)^{1/2}}{c_\varepsilon} C \partial_x u_{\varepsilon i}(0), \qquad i = 2, 3 \tag{2.53}$$

$$\frac{T_\varepsilon(0) - T_1}{T_1} = \varepsilon \lambda(T_1) \frac{2E}{5c_\varepsilon (RT_1)^{1/2}} \partial_x T_\varepsilon(0) \tag{2.54}$$

and at $x = L$,

$$u_{\varepsilon i}(L) - u_{2i} = -\varepsilon \mu(T_2) \frac{(RT_2)^{1/2}}{c_\varepsilon} C \partial_x u_{\varepsilon i}(L), \qquad i = 2, 3 \tag{2.55}$$

$$\frac{T_\varepsilon(L) - T_2}{T_2} = -\varepsilon \lambda(T_2) \frac{2E}{5c_\varepsilon (RT_2)^{1/2}} \partial_x T_\varepsilon(L) \tag{2.56}$$

We assume that the constants E, C are positive (see the approximate computation of these constants in ref. 14). Since the viscosity and the thermal conductivity satisfy (2.26)–(2.27), we get from (2.52)

$$T_\varepsilon(x) = (a_\varepsilon x + b_\varepsilon)^{1/(\alpha + 1)} \tag{2.57}$$

Finally, we want the Chapman–Enskog expansion to be an approximate solution of (2.5); thus,

$$\int \rho_\varepsilon(x)\, dx = N$$

and therefore the constant c_ε in (2.50) satisfies

$$c_\varepsilon = N \Big/ \int \frac{dx}{RT_\varepsilon} \tag{2.58}$$

For $\varepsilon = 0$, (2.54) and (2.56)–(2.58) have a unique solution defined by the parameters

$$a_0 = \frac{T_2^{\alpha+1} - T_1^{\alpha+1}}{L} \tag{2.59}$$

$$b_0 = T_1^{\alpha+1} \tag{2.60}$$

$$c_0 = \frac{\alpha}{\alpha+1} \frac{NR}{L} \frac{T_2^{\alpha+1} - T_1^{\alpha+1}}{T_2^\alpha - T_1^\alpha} \tag{2.61}$$

Using the implicit function theorem, the system (2.54), (2.56)–(2.58) has a unique solution for ε small enough which is of class C^1 with respect to ε and, using (2.27), such that

$$\frac{\partial a_\varepsilon}{\partial \varepsilon}(\varepsilon = 0) = -\frac{\alpha+1}{\alpha} \frac{2EC_\lambda}{5NRL} (T_2^\alpha - T_1^\alpha) \left(\frac{T_2^{\alpha+1}}{(RT_2)^{1/2}} + \frac{T_1^{\alpha+1}}{(RT_1)^{1/2}} \right) \tag{2.62}$$

$$\frac{\partial b_\varepsilon}{\partial \varepsilon}(\varepsilon = 0) = \frac{\alpha+1}{\alpha} \frac{2EC_\lambda}{5NR} (T_2^\alpha - T_1^\alpha) \frac{T_1^{\alpha+1}}{(RT_1)^{1/2}} \tag{2.63}$$

$$\begin{aligned} \frac{\partial c_\varepsilon}{\partial \varepsilon}(\varepsilon = 0) = {} & -\frac{2EC_\lambda}{5L} \left(\frac{T_2^{\alpha+1}}{(RT_2)^{1/2}} + \frac{T_1^{\alpha+1}}{(RT_1)^{1/2}} \right) \\ & + \frac{2EC_\lambda}{5L} \frac{\alpha}{\alpha+1} \frac{T_2^{\alpha+1} - T_1^{\alpha+1}}{T_2^\alpha - T_1^\alpha} \left(\frac{T_2^\alpha}{(RT_2)^{1/2}} + \frac{T_1^\alpha}{(RT_1)^{1/2}} \right) \end{aligned} \tag{2.64}$$

From (2.51) and (2.57)

$$u_{\varepsilon i}(x) = d_{\varepsilon i}(a_\varepsilon x + b_\varepsilon)^{1/(\alpha+1)} + e_{\varepsilon i}, \qquad i = 2, 3 \tag{2.65}$$

where $d_{\varepsilon i}$, $e_{\varepsilon i}$ are constants whose values are computed from (2.53), (2.55). This concludes the proof of Property 2.4.1.

The implicit function theorem was used because (2.58) is a nonlocal condition. Without this condition, the existence of a solution for the

Navier–Stokes system with the slip boundary conditions could be proved for any ε using simple estimates and nonlinear theory.

Note that in this case the Euler system is reduced to Eqs. (2.49)–(2.50) and so is not well posed (this is due to the fact that $u_{\varepsilon 1}$ is equal to 0).

2.5. Justification of the Chapman–Enskog Expansion and the Boundary Conditions

Let

$$F(x, \xi) = F_{\mathrm{CE}}(x, \xi) + \varepsilon^2 W(x, \xi) + \chi_1\left(\frac{x}{\varepsilon}, \xi\right) + \chi_2\left(\frac{L-x}{\varepsilon}, \xi\right) \quad (2.66)$$

We prove in this section that if ρ_ε, u_ε, T_ε satisfy the Navier–Stokes system (2.16)–(2.19) and the boundary conditions (2.37)–(2.39), then $F(x, \xi)$ given by (2.66) is an approximate solution of (2.1)–(2.5) [with α_1 given by (2.40)].

2.5.1. Estimate for the Kinetic Boundary Layer Terms: χ_1, χ_2. First recall the result of ref. 2.

Lemma 2.5.1. Let ϕ be a function such that

$$\int (1 + |\xi|)\, M_0^{-1}(\xi)(\phi(\xi))^2\, d\xi < +\infty$$

Then there exist a unique solution χ of (2.34)–(2.36) in $L^\infty(dx, L^2((1+|\xi|)\,d\xi))$ and a unique asymptotic limit $\chi_\infty = [a + b_2\xi_2 + b_3 c_3 + c(|\xi|^2 - 3)/2] M_0$ such that

$$\int (1 + |\xi|)\, M_0^{-1}(\chi(x, \xi) - \chi_\infty)^2\, d\xi \leqslant C e^{-\gamma x} \int (1 + |\xi|)\, M_0^{-1}\phi^2\, d\xi$$

where C, γ are positive constants.

According to this lemma, let χ_1 satisfy (2.30), (2.31) with zero mass flux and α_1 given by (2.40). From we obtain (2.31), (2.46), and (2.47)

$$\int_{\xi_1 > 0} (1 + |\xi|)\, M_{u_1, T_1}^{-1}(\chi_1(0, \xi))^2\, d\xi \leqslant Cst \cdot \varepsilon^2$$

Moreover, the boundary conditions (2.37)–(2.40) ensure that the limit of χ_1 when x tends to infinity is zero, whence, according to Lemma 2.5.1,

$$\int (1 + |\xi|)(\chi_1(x, \xi))^2\, M_{u_1, T_1}^{-1}\, d\xi \leqslant C_1 e^{-\gamma_1 x} \varepsilon^2 \quad (2.67)$$

where C_1 and γ_1 are positive constants depending only on u_1 and T_1. We get a similar estimate for χ_2.

2.5.2. Estimate for W and $\partial_x W$. Let us first recall two basic properties of the Boltzmann operator.[15]

Let M be a Maxwellian; then there exist two positive constants C_M, C'_M depending only on the density, velocity, and temperature of the Maxwellian M such that:

(1) If f satisfies

$$\int \psi(\xi) f(\xi)\, d\xi = 0 \qquad \text{for} \quad \psi(\xi) = 1, \xi_1, \xi_2, \xi_3, |\xi|^2 \tag{2.68}$$

then

$$C_M \int (1+|\xi|)\, M^{-1} f_2\, d\xi \leqslant \int M^{-1} Q(M, f) f\, d\xi \tag{2.69}$$

(2) For any functions f, g

$$\int (1+|\xi|)^{-1} M^{-1} (Q(f, g))^2\, d\xi \leqslant C'_M \left[\int (1+|\xi|)\, M^{-1} f^2\, d\xi\right]\left[\int (1+|\xi|)\, M^{-1} g^2\, d\xi\right] \tag{2.70}$$

Using both properties, we obtain the following estimates for W and $\partial_x W$.

Lemma 2.5.2. There exist two constants C_W, C'_W independent of ε such that

$$\int (1+|\xi|)\, M^{-1}_{u_\varepsilon, T_\varepsilon} W^2\, d\xi \leqslant C_W \tag{2.71}$$

$$\int (1+|\xi|)\, M^{-1}_{u_\varepsilon, T_\varepsilon} (\partial_x W)^2\, d\xi \leqslant C'_W \tag{2.72}$$

Proof. Estimate for W. Multiplying (2.23) by $M^{-1}_{u_\varepsilon, T_\varepsilon} W$ using (2.22) and property (2.69), we get

$$Cst \cdot \rho_\varepsilon \int (1+|\xi|)\, M^{-1}_{u_\varepsilon, T_\varepsilon} W^2\, d\xi \leqslant \left[\int (1+|\xi|)^{-1} M^{-1}_{u_\varepsilon, T_\varepsilon} S_\varepsilon^2\, d\xi\right]^{1/2} \left[\int (1+|\xi|)\, M^{-1}_{u_\varepsilon, T_\varepsilon} W^2\, d\xi\right]^{1/2} \tag{2.73}$$

The constant Cst does not depend on ε. By (2.70), we estimate $Q(M_{u_\varepsilon, T_\varepsilon} h_\varepsilon, M_{u_\varepsilon, T_\varepsilon} h_\varepsilon)$ in term of h_ε. Notice that

$$\partial_x (M_{u_\varepsilon, T_\varepsilon}) = \left(\frac{\xi - u_\varepsilon}{RT_\varepsilon}\, \partial_x u_\varepsilon + \frac{|\xi - u_\varepsilon|^2 - 3RT_\varepsilon}{2RT_\varepsilon}\, \partial_x \log(T_\varepsilon)\right) M_{u_\varepsilon, T_\varepsilon} \tag{2.74}$$

Using Property 2.4.1, the derivatives at any order of the fluid quantities ρ_ε, u_ε, T_ε remain bounded independently of ε. Thus (2.71) is proved.

Estimate for $\partial_x W$. Differentiating (2.22), (2.23), we obtain

$$2Q(\rho_\varepsilon M_{u_\varepsilon, T_\varepsilon}, \partial_x W) = -2Q(\partial_x(\rho_\varepsilon M_{u_\varepsilon, T_\varepsilon}), W) + \partial_x S_\varepsilon \tag{2.75}$$

$$\forall x \in [0, L], \qquad \int \psi(\xi)\, \partial_x W(x, \xi)\, d\xi = 0 \qquad \text{for} \quad \psi(\xi) = 1, \xi_1, \xi_2, \xi_3, |\xi|^2 \tag{2.76}$$

Multiply (2.75) by $M^{-1}_{u_\varepsilon, T_\varepsilon} \partial_x W$, using (2.69),

$$\begin{aligned}
&Cst \cdot \rho_\varepsilon \int (1 + |\xi|)\, M^{-1}_{u_\varepsilon, T_\varepsilon} (\partial_x W)^2\, d\xi \\
&\leqslant \left\{ \int (1 + |\xi|)^{-1} M^{-1}_{u_\varepsilon, T_\varepsilon} [2Q(\partial_x(\rho_\varepsilon M_{u_\varepsilon, T_\varepsilon}), W)]^2\, d\xi \right\}^{1/2} \\
&\quad \times \left[\int (1 + |\xi|)\, M^{-1}_{u_\varepsilon, T_\varepsilon} (\partial_x W)^2\, d\xi \right]^{1/2} \\
&\quad + \left[\int (1 + |\xi|)^{-1}\, \mathrm{M}^{-1}_{u_\varepsilon, T_\varepsilon} (\partial_x S_\varepsilon)^2\, d\xi \right]^{1/2} \left[\int (1 + |\xi|)\, M^{-1}_{u_\varepsilon, T_\varepsilon} (\partial_x W)^2\, d\xi \right]^{1/2}
\end{aligned}$$

The proof of (2.72) is then obtained using the estimate for W and (2.70).

2.5.3. Estimate in the Slab: [0, *L*]. Notice that the estimate for χ_1 uses M_{u_1, T_1}, that for χ_2 uses M_{u_2, T_2}, and the estimates for W and $\partial_x W$ use $M_{u_\varepsilon, T_\varepsilon}$. In order to have a simple estimation of the approximate, we need a "single Maxwellian." Assume that $T_1 \leqslant T_2$, let $T > T_2$, and let M be the Maxwellian of temperature T and mean velocity 0. Then there exist constants such that

$$M_{u_1, T_1}(\xi) \leqslant C_1 M(\xi), \quad M_{u_2, T_2}(\xi) \leqslant C_2 M(\xi), \quad M_{u_\varepsilon, T_\varepsilon}(\xi) \leqslant C_3 M(\xi), \quad \forall \xi, \forall x$$

From (2.67), (2.71), and (2.72) we thus have

$$\int (1 + |\xi|)[\chi_1(x, \xi)]^2 M^{-1}\, d\xi \leqslant Cst \cdot \varepsilon^2 e^{-\gamma_1 x} \tag{2.77}$$

$$\int (1 + |\xi|)\, M^{-1} W^2\, d\xi \leqslant Cst \tag{2.78}$$

$$\int (1 + |\xi|)\, M^{-1} (\partial_x W)^2\, d\xi \leqslant Cst \tag{2.79}$$

By the very definitions of $F, h_\varepsilon, W, \chi_1, \chi_2$

$$\begin{aligned}\xi_1 \partial_x F - \frac{1}{\varepsilon} Q(F, F) = {} & \varepsilon^2(\xi_1 \partial_x W - 2Q(M_{u_\varepsilon, T_\varepsilon} h_\varepsilon, W) - \varepsilon Q(W, W)) \\ & - \frac{2}{\varepsilon} Q(M_{u_\varepsilon, T_\varepsilon} - M_{u_1, T_1}, \chi_1) - 2Q(M_{u_\varepsilon, T_\varepsilon} h_\varepsilon, \chi_1) \\ & - 2\varepsilon Q(W, \chi_1) - \frac{1}{\varepsilon} Q(\chi_1, \chi_1) \\ & - \frac{2}{\varepsilon} Q(M_{u_\varepsilon, T_\varepsilon} - M_{u_2, T_2}, \chi_2) - 2Q(M_{u_\varepsilon, T_\varepsilon} h_\varepsilon, \chi_2) \\ & - 2\varepsilon Q(W, \chi_2) - \frac{1}{\varepsilon} Q(\chi_2, \chi_2) - \frac{2}{\varepsilon} Q(\chi_1, \chi_2) \end{aligned} \tag{2.80}$$

Using (2.67)–(2.70) and Lemma 2.5.2,

$$\int (1 + |\xi|)^{-1} M^{-1}(Q(M_{u_\varepsilon, T_\varepsilon} h_\varepsilon, W))^2 \, d\xi \leqslant Cst$$

$$\int (1 + |\xi|)^{-1} M^{-1}(Q(W, \chi_1))^2 \, d\xi \leqslant Cst \cdot \varepsilon^2$$

$$\int (1 + |\xi|)^{-1} M^{-1}(Q(\chi_1, \chi_1))^2 \, d\xi \leqslant Cst \cdot \varepsilon^2 e^{-2\gamma_1 x/\varepsilon}, \qquad \forall x \in [0, L]$$

$$\int (1 + |\xi|)^{-1} M^{-1}(Q(\chi_1, \chi_2))^2 \, d\xi \leqslant Cst \cdot \varepsilon^2 e^{-2\gamma_1 x/\varepsilon} e^{-2\gamma_2 (L-x)/\varepsilon}, \qquad \forall x \in [0, L]$$

$$\int (1 + |\xi|)^{-1} M^{-1}(Q(M_{u_\varepsilon, T_\varepsilon} h, \chi_1))^2 \, d\xi \leqslant Cst \cdot \varepsilon^2 e^{-2\gamma_1 x/\varepsilon}$$

$$\int (1 + |\xi|)^{-1} M^{-1}(Q(M_{u_\varepsilon, T_\varepsilon} - M_1, \chi_1))^2 \, d\xi \leqslant Cst \cdot (\varepsilon + Cst \cdot x)^2 \varepsilon^2 e^{-2\gamma_1 x/\varepsilon}$$

The same estimate holds for χ_2. Evaluating the $L^2([0, L])$ norm of these last terms and replacing in (2.80), we get the following property.

Property 2.5.1. There exists a constant Cst such that

$$\left\{ \int_0^L \int (1 + |\xi|)^{-1} M^{-1} \left[\xi_1 \partial_x F - \frac{1}{\varepsilon} Q(F, F) \right]^2 d\xi \, dx \right\}^{1/2} \leqslant Cst \cdot \varepsilon^{3/2} \tag{2.81}$$

2.5.4. Estimate at $x=0$, at $x=L$, and for the Total Number of Particles

Property 2.5.2. There exist constants Cst independent of ε such that the following estimates hold:

$$\left| \int (1+|\xi|)\, M^{-1}[F(0,\xi) - \alpha_1 M_{u_1, T_1}(\xi)]^2\, d\xi \right| \leqslant Cst \cdot \varepsilon^2 \tag{2.82}$$

[where α_1 is given by (2.40)] and similarly at $x=L$

$$\int \xi_1 F(x,\xi)\, d\xi = 0 \tag{2.83}$$

$$\left| \int_0^L \int F(x,\xi)\, d\xi\, dx - N \right| \leqslant Cst \cdot \varepsilon^2 \tag{2.84}$$

Proof. At $x=0$, the boundary layer term χ_1 has been constructed to correct the Chapman–Enskog expansion in order to fit the boundary condition (2.2) where α_1 is given by (2.40). The contributions at $x=0$ of the terms χ_2 and W are from (2.67) (with χ_2 instead of χ_1) and (2.71) of order ε^2. This proves (2.82).

From (2.49), we know that the total mass flux of $\rho_\varepsilon M_{u_\varepsilon, T_\varepsilon}$ is zero. In the same way, according to (2.22), (2.25), the total mass fluxes of W and $M_{u_\varepsilon, T_\varepsilon}(\xi)\, h_\varepsilon(\xi)$ are also zero. But χ_1 and χ_2 do not contribute to this total mass flux; (2.83) is thus proved.

By the construction (2.58), the total number of particles of $\rho_\varepsilon M_{u_\varepsilon, T_\varepsilon}$ is N. Moreover, according to (2.25), the density of $M_{u_\varepsilon, T_\varepsilon}(\xi)\, h_\varepsilon(\xi)$ is equal to zero. In the same way, from (2.22), we know that W does not contribute to the density of particles. According to (2.67),

$$\left| \int_0^L \int \chi_1\left(\frac{x}{\varepsilon}, \xi\right) d\xi\, dx \right| \leqslant Cst \cdot \varepsilon^2$$

and a similar estimate for χ_2. Thus, (2.84) is proved.

Properties (2.66), (2.67) prove that F is an approximate solution of (2.1)–(2.5).

Remark 2.5.1. The change of any coefficient (C_1, C_2, C_3) in the slip boundary conditions (2.38)–(2.39) induces an error in the fluid quantities ρ_ε, u_ε, T_ε of order ε. Estimations (2.81)–(2.84) prove that the error made in the resolution of the Navier–Stokes system by using, in the slip boundary conditions, the coefficients given by the above linear half-space analysis is of order less than $\varepsilon^{3/2}$. This shows that we have obtained good coefficients for the slip boundary conditions.

2.6. Noncomplete Accommodation

In Sections 2.1–2.5, complete accommodation at the boundary was assumed. The results obtained are easily extended for more complex wall–surface interactions. Let us, for example, consider Maxwell accommodation. Then (2.2) is replaced by

$$\exists\alpha_1 \in \mathbf{R}/F_\varepsilon(0, \xi) = \beta_1 F_\varepsilon(0, R\xi) + (1-\beta_1)\,\alpha_1 M_{u_1, T_1}(\xi), \qquad \xi_1 > 0 \tag{2.85}$$

The parameter β_1 is the Maxwell accommodation coefficient: β_1 is given in $[0, 1[$ and R is defined by

$$R(\xi_1, \xi_2, \xi_3) = (-\xi_1, \xi_2, \xi_3)$$

A similar equation holds at $x = L$; (2.3) becomes

$$\exists\alpha_2 \in \mathbf{R}/F_\varepsilon(L, \xi) = \beta_2 F_\varepsilon(L, R\xi) + (1-\beta_2)\,\alpha_2 M_{u_2, T_2}(\xi), \qquad \xi_1 < 0 \tag{2.86}$$

Equations (2.1), (2.4), and (2.5) remain unchanged.

The Chapman–Enskog expansion and the Navier–Stokes equations remain as in Section 2.2.

2.7. Derivation of the Slip Boundary Conditions

We add to $F_{\mathrm{CE}}(x, \xi)$ two kinetic boundary layer terms at $x = 0$ and at $x = L$,

$$\chi_1\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad \chi_2\left(\frac{L-x}{\varepsilon}, \xi\right)$$

As in Section 2.3, from (2.1), (2.85), (2.86), and (2.4), we obtain the following problem for χ_1:

$$\xi_1 \partial_x \chi_1 - 2Q(M_{u_1, T_1}, \chi_1) = 0, \qquad 0 \leqslant x < \infty \tag{2.87}$$

$$\begin{aligned} \chi_1(0, \xi) = {} & \beta_1 \chi_1(0, R\xi) + (1-\beta_1)\,\alpha_1 M_{u_1, T_1}(\xi) \\ & - F_{\mathrm{CE}}(0, \xi) + \beta_1 F_{\mathrm{CE}}(0, R\xi), \qquad \xi_1 > 0 \end{aligned} \tag{2.88}$$

This linearized problem has been studied by Coron *et al.*[10] and the results are very similar to the problem with incoming flux. We still want χ_1 to go to zero when $x \to +\infty$; this condition is equivalent to certain equalities on the Chapman–Enskog expansion at $x = 0$. To have the u_1, T_1 dependence of these conditions, we change ξ in $\bar{\xi}$ [see (2.11)] and obtain

$$\xi_1 \partial_x \chi - 2Q(M_0, \chi) = 0 \tag{2.89}$$

$$\chi(0, \xi) = \beta_1 \chi(0, R\xi) + \phi(\xi), \qquad \xi_1 > 0 \tag{2.90}$$

$$\int \xi_1 \chi(x, \xi) \, d\xi = 0 \tag{2.91}$$

for various functions ϕ.

We notice that if ϕ is odd (respectively even) with respect to ξ_2, then the limit also is odd (respectively even). Moreover, if ϕ is equal to M_0, $\xi_2 M_0$, $\xi_3 M_0$, or $\frac{1}{2}(|\xi|^2 - 3) M_0$, then the solution χ is constant and equal to $[1/(1-\beta_1)]\phi$. Since χ_1, like F_ε, has zero mass flux, we obtain

$$u_{\varepsilon 1}(0) = 0 \tag{2.92}$$

and χ_1 tends to zero at infinity if and only if the following boundary conditions hold:

$$\rho_\varepsilon(0)\, u_{\varepsilon i}(0) = \varepsilon C_{1\beta} \partial_x u_{\varepsilon i}(0), \qquad i = 2, 3 \tag{2.93}$$

$$\rho_\varepsilon(0)[T_\varepsilon(0) - T_1] = \varepsilon C_{2\beta} \partial_x u_{\varepsilon 1}(0) + \varepsilon C_{2\beta} \partial_x T_\varepsilon(0) \tag{2.94}$$

Moreover, as in Section 2.3, we obtain a relation which gives α_1,

$$\alpha_1 = \rho_\varepsilon(0) + \varepsilon C_{4\beta} \partial_x u_{\varepsilon 1}(0) + \varepsilon C_{5\beta} \partial_x T_\varepsilon(0) \tag{2.95}$$

$C_{1\beta}, C_{2\beta}, \ldots, C_{5\beta}$ are constants depending on T_1, $\beta = \beta_1$:

$$C_{1\beta} = \frac{\mu(T_1)}{(RT_1)^{1/2}} (1 + \beta_1) c_{1\beta}, \qquad C_{2\beta} = \frac{\mu(T_1)}{R} (1 - \beta_1) c_{2\beta}$$

$$C_{3\beta} = \frac{2\lambda(T_1)}{5R(RT_1)^{1/2}} (1 + \beta_1) c_{3\beta} \tag{2.96}$$

$$C_{4\beta} = \frac{\mu(T_1)}{R} (1 - \beta_1) c_{4\beta}, \qquad C_{5\beta} = \frac{2\lambda(T_1)}{5R(RT_1)^{1/2}} (1 + \beta_1) c_{5\beta} \tag{2.97}$$

$c_{1\beta}, c_{2\beta}, \ldots, c_{5\beta}$ are constants depending on $\beta = \beta_1$ given by the asymptotic limit of the solutions of (2.89)–(2.91) for the following functions $\phi(\xi)$:

$$\text{for} \quad \phi(\xi) = (\xi_1 \xi_2) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = c_{1\beta} \xi_2 M_0(\xi) \tag{2.98}$$

$$\text{for} \quad \phi(\xi) = \left(\xi_1^2 - \frac{|\xi|^2}{3}\right) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = \left(c_{4\beta} + c_{2\beta} \frac{|\xi|^2 - 3}{2}\right) M_0(\xi) \tag{2.99}$$

$$\text{for} \quad \phi(\xi) = \xi_1 \left(\frac{|\xi|^2 - 5}{2}\right) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = \left(c_{5\beta} + c_{3\beta} \frac{|\xi|^2 - 3}{2}\right) M_0(\xi) \tag{2.100}$$

As in Section 2.3, we get four scalar boundary conditions at $x=0$. Similar equations are derived at $x=L$.

With $\beta_1=0$ these boundary conditions become exactely the conditions obtained in Section 2.3.

The proof is achieved as in Sections 2.4, 2.5; Properties 2.5.1 and 2.5.2 are also true in the case of Maxwell accommodation.

Notice that the Maxwell condition (2.85) can be replaced by a more complex one.[7,10]

3. FORMAL DERIVATION OF SLIP BOUNDARY CONDITIONS

3.1. Introduction

Let us consider the general problem of a flow around a body Ω (see Fig. 4). As was done by Grad,[17] we are going to derive formally the slip boundary conditions for the Navier–Stokes system from the kinetic boundary conditions at the wall. As in the previous section, this derivation relies on the asymptotic behavior of the linear half-space problem.

The distribution of particles F_ε satisfies the Boltzmann equation

$$\xi \cdot \partial_x F_\varepsilon - \frac{1}{\varepsilon} Q(F_\varepsilon, F_\varepsilon) = 0, \qquad x \in \mathbf{R}^3 - \Omega, \qquad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 \quad (3.1)$$

There are some boundary conditions at infinity and we assume Maxwell accommodation at the wall. Denoting by $\partial\Omega = \Sigma$ the boundary of Ω, we have

$$\forall x \in \Sigma, \quad \exists \alpha(x) \in \mathbf{R} / F_\varepsilon(x, \xi) = \beta(x) F_\varepsilon(x, R\xi) + [1 - \beta(x)] \alpha(x) M_{0,T}(\xi), \qquad \xi \cdot n(x) > 0 \quad (3.2)$$

$$\forall x \in \Sigma, \quad \int \xi \cdot n(x) F_\varepsilon(x, \xi) \, d\xi = 0 \quad (3.3)$$

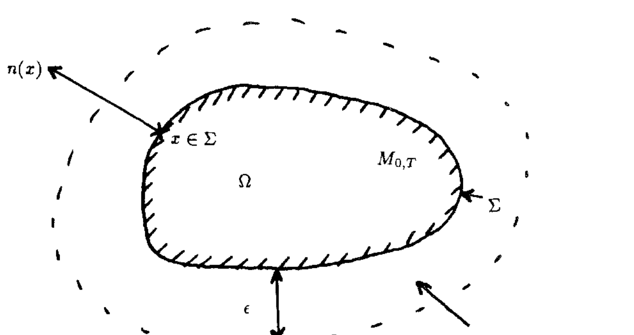

Fig. 4. Flow field.

where $n(x)$ is the outward normal of Ω at point $x \in \Sigma$, $\beta(x)$ is the Maxwell accommodation coefficient, and $R\xi = \xi - 2[\xi \cdot n(x)]\, n(x)$. The wall is supposed to be at rest and at temperature T at point $x \in \Sigma$. (Thus the Maxwellian of the wall, $M_{0,T}$, has no mean velocity).

3.2. Chapman–Enskog Expansion

As in the previous section, we first assume that outside a layer of thickness on the order of the mean free path, the Chapman–Enskog expansion is valid (see Section 4 for the discussion of the validity of this assumption).

The Chapman–Enskog expansion reads [see (2.8)]

$$F_{\mathrm{CE}}(x, \xi) = M_{u_\varepsilon, T_\varepsilon}(\xi)[\rho_\varepsilon - \varepsilon h_\varepsilon(x, \xi)] \tag{3.4}$$

where ρ_ε, u_ε, T_ε depend on x. The Maxwellian is given by (2.9) and $h_\varepsilon(x, \xi)$ is written in terms of Sonine polynomials $A_i(\bar{\xi})$ and $B_{i,j}(\bar{\xi})$ by

$$\begin{aligned} h_\varepsilon(x, \xi) = &\sum_{1 \leqslant i, j \leqslant 3} b(T_\varepsilon, |\bar{\xi}|)\, B_{i,j}(\bar{\xi})\, \partial_{x_i} u_{\varepsilon j} \\ &+ \sum_{i=1,2,3} a(T_\varepsilon, |\bar{\xi}|)\, A_i(\bar{\xi}) (RT_\varepsilon)^{1/2} \partial_{x_i} \log(T_\varepsilon) \\ = &\, b(T_\varepsilon, |\bar{\xi}|)\, B(\bar{\xi}) : \nabla u_\varepsilon + a(T_\varepsilon, |\bar{\xi}|)(RT_\varepsilon)^{1/2}\, A(\bar{\xi}) \cdot \nabla[\log(T_\varepsilon)] \end{aligned} \tag{3.5}$$

where $\bar{\xi}$ is given by (2.11) and

$$A_i(\bar{\xi}) = \bar{\xi}_i \frac{|\bar{\xi}|^2 - 5}{2}, \qquad i = 1, 2, 3 \tag{3.6}$$

$$B_{i,j}(\bar{\xi}) = \bar{\xi}_i \bar{\xi}_j - \frac{|\bar{\xi}|^2}{3} \delta_{i,j}, \qquad 1 \leqslant i, j \leqslant 3 \tag{3.7}$$

$a(T_\varepsilon, |\bar{\xi}|)$ and $b(T_\varepsilon, |\bar{\xi}|)$ satisfy

$$-2Q(M_{u_\varepsilon, T_\varepsilon}, M_{u_\varepsilon, T_\varepsilon} a(T_\varepsilon, |\bar{\xi}|)\, A_i(\bar{\xi})) = A_i(\bar{\xi}) M_{u_\varepsilon, T_\varepsilon}, \qquad i = 1, 2, 3 \tag{3.8}$$

$$-2Q(M_{u_\varepsilon, T_\varepsilon}, M_{u_\varepsilon, T_\varepsilon}\, b(T_\varepsilon, |\bar{\xi}|)\, B_{i,j}(\bar{\xi})) = B_{i,j}(\bar{\xi}) M_{u_\varepsilon, T_\varepsilon}, \qquad 1 \leqslant i, j \leqslant 3 \tag{3.9}$$

The stationary Navier–Stokes system is

$$\operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \tag{3.10}$$

$$\sum_{j=1,2,3} u_{\varepsilon j}\partial_{x_j}u_{\varepsilon i}+\frac{1}{\rho_\varepsilon}\partial_{x_i}(\rho_\varepsilon RT_\varepsilon) \tag{3.11}$$

$$=\varepsilon\frac{1}{\rho_\varepsilon}\sum_{j=1,2,3}\partial_{x_j}\left\{\mu(T_\varepsilon)\left[\partial_{x_j}u_{\varepsilon i}+\partial_{x_i}u_{\varepsilon j}-\frac{2}{3}\operatorname{div}(u_\varepsilon)\delta_{i,j}\right]\right\},\qquad i=1,2,3$$

$$u_\varepsilon\cdot\operatorname{grad}(T_\varepsilon)+\frac{2}{3}T_\varepsilon\operatorname{div}(u_\varepsilon)=\varepsilon\frac{2}{3R}\frac{1}{\rho_\varepsilon}\{\operatorname{div}[\lambda(T_\varepsilon)\operatorname{grad}(T_\varepsilon)]+\psi_\varepsilon\} \tag{3.12}$$

with

$$\psi_\varepsilon=\mu(T_\varepsilon)\left[\frac{1}{2}\sum_{1\leqslant i,j\leqslant 3}(\partial_{x_i}u_{\varepsilon j}+\partial_{x_j}u_{\varepsilon i})^2-\frac{2}{3}(\operatorname{div}u_\varepsilon)^2\right] \tag{3.13}$$

where the thermal conductivity $\lambda(T_\varepsilon)$ and $\mu(T_\varepsilon)$ are still expressed in terms of $a(T_\varepsilon,\xi)$ and $b(T_\varepsilon,\xi)$ by (2.20), (2.21). As in Section 2.2, we make the approximation that $a(T_\varepsilon,\xi)$ and $b(T_\varepsilon,\xi)$ do not depend on ξ. This assumption is only made to get simpler results (see Section 2.2). In particular, the Chapman–Enskog expansion is then written in terms of $\lambda(T_\varepsilon)$ and $\mu(T_\varepsilon)$ instead of $a(T_\varepsilon,\xi)$ and $b(T_\varepsilon,\xi)$,

$$h_\varepsilon(x,\xi)=\sum_{1\leqslant i,j\leqslant 3}\frac{\mu(T_\varepsilon)}{(RT_\varepsilon)^2}\left[(\xi_i-u_{\varepsilon i})(\xi_j-u_{\varepsilon j})-\frac{|\xi-u_\varepsilon|^2}{3}\delta_{i,j}\right]\partial_{x_i}u_{\varepsilon j}$$
$$+\sum_{1\leqslant i\leqslant 3}\frac{2\lambda(T_\varepsilon)}{5(RT_\varepsilon)^2}(\xi_i-u_{\varepsilon i})\frac{|\xi-u_\varepsilon|^2-5RT_\varepsilon}{2RT_\varepsilon}\partial_{x_i}T_\varepsilon$$

Let $\rho_\varepsilon, u_\varepsilon, T_\varepsilon$ satisfy the Navier–Stokes system (3.10)–(3.12); then there exists a function $W(x,\xi)$ such that

$$\forall x\in\Omega,\qquad \int\psi(\xi)\,W(x,\xi)\,d\xi=0\qquad\text{for}\quad \psi(\xi)=1,\xi_1,\xi_2,\xi_3,|\xi|^2 \tag{3.14}$$

and

$$-2Q(\rho_\varepsilon M_{u_\varepsilon,T_\varepsilon},W)=S_\varepsilon \tag{3.15}$$

with

$$S_\varepsilon(x,\xi)=\xi\cdot\partial_x(M_{u_\varepsilon,T_\varepsilon}h_\varepsilon)+Q(M_{u_\varepsilon,T_\varepsilon}h_\varepsilon,M_{u_\varepsilon,T_\varepsilon}h_\varepsilon)$$
$$-\sum_{i=1,2,3}\left\{\frac{\xi_i-u_{\varepsilon i}}{RT_\varepsilon}\sum_{j=1,2,3}\partial_{x_j}\left[\mu(T_\varepsilon)(\partial_{x_j}u_{\varepsilon i}+\partial_{x_i}u_{\varepsilon j})\right.\right.$$
$$\left.\left.-\frac{2}{3}\operatorname{div}(u_\varepsilon)\delta_{i,j}\right]\right\}M_{u_\varepsilon,T_\varepsilon}(x,\xi)$$
$$-\frac{2}{3}\frac{1}{RT_\varepsilon}\frac{|\xi-u_\varepsilon|^2-3RT_\varepsilon}{2RT_\varepsilon}\{\operatorname{div}[\lambda(T_\varepsilon)\operatorname{grad}(T_\varepsilon)]+\psi_\varepsilon\}M_{u_\varepsilon,T_\varepsilon}(x,\xi)$$

(see ref.1 for the proof) and thus

$$\xi \cdot \partial_x(F_{\rm CE} + \varepsilon^2 W) - \frac{1}{\varepsilon} Q(F_{\rm CE} + \varepsilon^2 W, F_{\rm CE} + \varepsilon^2 W)$$
$$= \varepsilon^2 [\xi \cdot \partial_x W - 2Q(M_{u_\varepsilon, T_\varepsilon} h_\varepsilon, W) - \varepsilon Q(W, W)] \qquad (3.16)$$

3.3. Derivation of the Slip Boundary Conditions for the Navier–Stokes Equations

Now we derive the boundary conditions for the Navier–Stokes equations. As in Section 2, we add to the Chapman–Enskog expansion a kinetic boundary layer term in order to take into account the boundary conditions imposed on F_ε. This kinetic layer term is going to be concentrated at the boundary of Ω. Let us first introduce the "boundary coordinates." As in ref. 3, we set

$$\Gamma_\delta = \{x \in \mathbf{R}^3 - \Omega / d(x, \Sigma) \leqslant \delta\}$$

Since Σ is at least C^2, we may define a function $\sigma(x)$, for $x \in \Gamma_\delta$ (δ small enough), such that

$$x = \sigma(x) + d(x, \Sigma)\, n(\sigma(x)), \qquad \sigma(x) \in \Sigma$$

[$n(\sigma(x))$ denotes the outward unit normal to Ω at $\sigma(x) \in \partial\Omega = \Sigma$]. Moreover, choosing δ small enough, we have

$$d(x) = d(x, \Sigma) \in C^2(\Gamma_\delta), \qquad \sigma(x) \in C^2(\Gamma_\delta)$$

We now solve for $x \in \Sigma$:

$$[\xi \cdot n(x)]\, \partial_\eta \chi^x(\eta, \xi) - 2Q(M_{0,T_i}, \chi^x(\eta, \xi)) = 0, \qquad 0 < \eta < +\infty, \qquad \xi \in \mathbf{R}^3 \qquad (3.17)$$

$$\chi^x(0, \xi) = \beta(x)\, \chi^x(0, R\xi) + [1 - \beta(x)]\, \alpha(x)\, M_{0,T}(\xi) - F_{\rm CE}(x, \xi),$$
$$\xi \cdot n(x) < 0 \qquad (3.18)$$

$$\int \xi \cdot n(x)\, \chi^x(\eta, \xi)\, d\xi = 0 \qquad (3.19)$$

For $x \in \Gamma_\delta$, we set

$$\chi(x, \xi) = \chi^{\sigma(x)}\left(\frac{d(x)}{\varepsilon}, \xi\right) \qquad (3.20)$$

We add to the Chapman–Enskog expansion this boundary layer term. As in Section 2, the boundary conditions for the Navier–Stokes system are derived from the fact that $\chi^x(\eta, \xi)$ should vanish as η goes to infinity.

We thus obtain for all $x \in \Sigma$

$$u_\varepsilon(x) \cdot n = 0 \tag{3.21}$$

$$\rho_\varepsilon(x)\, u_\varepsilon(x) \cdot \tau = \varepsilon C_{1\beta} \partial_n (u_\varepsilon \cdot \tau) + \varepsilon C_{6\beta} \partial_\tau T_\varepsilon \tag{3.22}$$

$$\rho_\varepsilon(x)[T_\varepsilon(x) - T] = \varepsilon C_{2\beta} \partial_n (u_\varepsilon \cdot n) + \varepsilon C_{3\beta} \partial_n T_\varepsilon + \varepsilon C_{7\beta} \operatorname{div}[u_\varepsilon - (u_\varepsilon \cdot n) n] \tag{3.23}$$

where n is the outward unit normal to Ω at point x and τ is any unit vector normal to n. We thus get four scalar equations. We have used the fact that $u_\varepsilon \cdot n = 0$ implies that $\partial_\tau (u_\varepsilon \cdot n) = 0$.

Moreover, as in Section 2, we obtain a relation which gives $\alpha(x)$,

$$\alpha(x) = \rho_\varepsilon(x) + \varepsilon C_{4\beta} \partial_n (u_\varepsilon \cdot n) + \varepsilon C_{5\beta} \partial_n T_\varepsilon + \varepsilon C_{8\beta} \operatorname{div}[u_\varepsilon - (u_\varepsilon \cdot n) n] \tag{3.24}$$

$C_{1\beta}, C_{2\beta},..., C_{8\beta}$ are constants depending on T, $\beta(x)$ and are given by

$$C_{1\beta} = \frac{\mu(T)}{(RT)^{1/2}} [1 + \beta(x)] c_{1\beta}, \qquad C_{2\beta} = \frac{\mu(T)}{R} [1 - \beta(x)] c_{2\beta}$$

$$C_{3\beta} = \frac{2\lambda(T)}{5R(RT)^{1/2}} [1 + \beta(x)] c_{3\beta} \tag{3.25}$$

$$C_{4\beta} = \frac{\mu(T)}{R} [1 - \beta(x)] c_{4\beta}, \qquad C_{5\beta} = \frac{2\lambda(T)}{5R(RT)^{1/2}} [1 + \beta(x)] c_{5\beta} \tag{3.26}$$

$$C_{6\beta} = \frac{2\lambda(T)}{5R(RT)^{1/2}} [1 - \beta(x)] c_{6\beta}, \qquad C_{7\beta} = \frac{\mu(T)}{R} [1 - \beta(x)] c_{7\beta}$$

$$C_{8\beta} = \frac{\mu(T)}{R} [1 - \beta(x)] c_{8\beta} \tag{3.27}$$

$c_{1\beta}, c_{2\beta},..., c_{8\beta}$ are constants depending on $\beta = \beta(x)$ given by the asymptotic limit of the solutions of (2.89)–(2.91) with $\beta_1 = \beta(x)$ and for the following functions $\phi(\xi)$:

$$\text{for} \quad \phi(\xi) = (\xi_1 \xi_2) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = c_{1\beta} \xi_2 M_0(\xi) \tag{3.28}$$

$$\text{for} \quad \phi(\xi) = \left(\xi_1^2 - \frac{|\xi|^2}{3} \right) M_0(\xi); \qquad \lim_{x \to \infty} \chi(x, \xi) = \left(c_{4\beta} + c_{2\beta} \frac{|\xi|^2 - 3}{2} \right) M_0(\xi) \tag{3.29}$$

$$\text{for}\quad \phi(\xi)=\xi_1\left(\frac{|\xi|^2-5}{2}\right)M_0(\xi);\quad \lim_{x\to\infty}\chi(x,\xi)=\left(c_{5\beta}+c_{3\beta}\frac{|\xi|^2-3}{2}\right)M_0(\xi) \tag{3.30}$$

$$\text{for}\quad \phi(\xi)=\xi_2\left(\frac{|\xi|^2-5}{2}\right)M_0(\xi);\quad \lim_{x\to\infty}\chi(x,\xi)=c_{6\beta}\xi_2 M_0(\xi) \tag{3.31}$$

$$\text{for}\quad \phi(\xi)=\left(\xi_2^2-\frac{|\xi|^2}{3}\right)M_0(\xi);\quad \lim_{x\to\infty}\chi(x,\xi)=\left(c_{8\beta}+c_{7\beta}\frac{|\xi|^2-3}{2}\right)M_0(\xi) \tag{3.32}$$

These results extend those obtained in Section 2.6. We have found the usual slip boundary conditions.[17,20] Note that in most current problems, the tangential derivatives as well as the term $C_{2\beta}\partial_n(u_\varepsilon\cdot n)$ are small compared with the normal derivatives of other quantities.[20] If we neglect these terms, the slip boundary conditions (3.22)–(3.23) become

$$\rho_\varepsilon(x)\,u_\varepsilon(x)\cdot\tau=\varepsilon C_{1\beta}\partial_n(u_\varepsilon\cdot\tau) \tag{3.22'}$$

$$\rho_\varepsilon(x)[T_\varepsilon(x)-T]=\varepsilon C_{3\beta}\partial_n T_\varepsilon \tag{3.23'}$$

Remark. In the previous derivation, we assumed that the wall's temperature was given. If this temperature is unknown, we have to add to the kinetic boundary conditions (3.2)–(3.3) another relation, for example, on the heat flux through the wall. Suppose we assume the following "adiabatic" condition at the kinetic level:

$$\int(\xi\cdot n)\,|\xi|^2\,F_\varepsilon(x,\xi)\,d\xi=0 \tag{3.33}$$

Then, the same kind of analysis can be made and we find the same slip boundary conditions with (3.23) being replaced by the usual adiabatic condition at the macroscopic level[16]

$$\lambda(T_\varepsilon)\,\partial_n T_\varepsilon+\mu(T_\varepsilon)\partial_n(|u_\varepsilon|^2/2)=0 \tag{3.34}$$

Note that this equation does not use any asymptotic limit of the half-space problem.

It could also be possible to take into account more complex relations than (3.33).

4. REMARKS ON THE VALIDITY OF THIS DERIVATION OF THE SLIP COEFFICIENTS

4.1. Estimate of the Solution

As in Section 2, it is possible to obtain formally that if the Navier–Stokes solution satisfies the slip boundary conditions (3.31)–(3.33), then there exists a kinetic layer term $\tilde{\chi}(x, \xi)$ concentrated near the wall such that the expansion

$$F(x, \xi) = F_{\text{CE}}(x, \xi) + \varepsilon^2 W(x, \xi) + \tilde{\chi}(x, \xi)$$

satisfies the boundary conditions (3.2)–(3.3) at order ε^2. We also formally get the following estimation:

$$\left\{\int_{\mathbf{R}^3 - \Omega} \int (1 + |\xi|)^{-1} M^{-1} \left[\xi \cdot \partial_x F - \frac{1}{\varepsilon} Q(F, F)\right]^2 d\xi\, dx\right\}^{1/2} \leqslant Cst \cdot \varepsilon^{3/2}$$

But because of the nonlinearity of the Boltzmann equation, it is not possible to conclude that the expansion built, F, gives an approximation of the Boltzmann solution at order ε. This is due to the fact that using the Chapman–Enskog expansion, we approximate the Boltzmann equation and not its solution.

4.2. Expansions Different from the Chapman–Enskog Theory

The above analysis was based on the Chapman–Enskog theory and the Navier–Stokes equations. However, in some cases, this expansion does not seem to be right approximation at order ε.

Darrozes[(11)] obtained that near the body, in a region which he called the boundary later, the macroscopic quantities satisfy the Navier–Stokes equations corrected by an additional term.

Studying the problem of thermal conduction around heated bodies, Wakabayashi and Sone[(30)] proved that the Navier–Stokes equation for the velocity needs to be corrected by adding the same kind of term as in ref. 11. This is due to the fact that the mean velocity has no zeroth-order term in ε.

However, in both cases, the theory of the half-space problem is related to the study of the kinetic layer term and gives the slip boundary conditions for the unknown of the expansion valid outside the Knudsen layer. Sone gives the framework of this derivation using two different expansions (see refs. 27 and 28, for example).

4.3. The Case of the Chapman–Enskog Expansion

In the Couette flow, the solutions of the Navier–Stokes system behave nicely when the Knudsen number tends to zero; $\rho_\varepsilon, u_\varepsilon, T_\varepsilon$ converge uniformly in the computational domain Ω to a regular profile when ε tends to zero. This is not true for general flow field. Indeed, the Reynolds number is defined by

$$Re = \rho VL/\varepsilon\mu$$

where V and L are a characteristic velocity and dimension of the flow (for example, the velocity at infinity and the length of one body). This Reynolds number goes to infinity when ε goes to 0. It seems, however, that the hypotheses used in the above sections for the derivation of the slip boundary conditions remain physically relevant. We are now trying to justify this affirmation in the two following cases: (1) when the flow is everywhere laminar, (2) when the turbulence is fully developed.

4.3.1. Laminar Regime. Near each body, there exists a laminar boundary layer the thickness of which is of order

$$\delta = L/(Re)^{1/2}$$

and thus is proportional to $\sqrt{\varepsilon}$ (see ref. 20, for example). The thickness of the Knudsen layer is of some mean free paths and thus is proportional to ε. So when ε is small, the Knudsen layer is much thinner than the viscous boundary layer. The thickness of the Knudsen layer justifies the one-dimensional analysis of the kinetic boundary layer term (see the scaling made in the normal direction of the boundary in the above sections). A solution of the Navier–Stokes problem has significant variations over distances normal to the boundary of order δ and so is approximately constant in the Knudsen layer, whose thickness is negligible compared to δ when ε is small. Moreover, the derivatives of fluid quantities are of order $1/\delta$; so if we assume the slip boundary condition found in the above sections, the velocity and the temperature of the Navier–Stokes solution in the Knudsen layer are close to those of the wall. This proves that the linear analysis of the kinetic layer term was indeed a good approach. Notice that outside the Knudsen layer and inside the viscous layer, the Chapman–Enskog expansion is not obviously false despite the rapid variation of the fluid quantities because they vary on a distance much longer than the mean free path.

4.3.2. Fully Developed Turbulence. When ε goes to zero, the Reynolds number goes to infinity and the flow regime becomes turbulent. However, we first examined the laminar case because the slip boundary conditions are usually used when the Knudsen number is small enough for the Navier–Stokes system to be valid but not too small to have slip effects (see the computation of flow using Navier–Stokes with slip boundary conditions done by Rostand[26]).

When the Reynolds number is large, the turbulence is fully developed. We assume that the Mach number is small and, therefore, that the incompressible Navier–Stokes system is a good approximation. In the case of fully developed turbulence, the macroscopic quantities vary over a distance much smaller than L. Let us look at the effect of fully developed turbulence on the derivation of slip boundary conditions. Frisch[13] shows how the breakdown of the hydrodynamic approximation seems to be unlike. We first recall some arguments developed in ref. 13. Let δ be the smallest scale of the flow. According to Kolmogorov,[21]

$$\delta \sim L \cdot (Re)^{-3/4}$$

Thus, δ behaves like $\varepsilon^{3/4}$ when ε is small. Since the mean free path l is proportional to ε, it is much larger than δ (except if the Kolmogorov exponential law is far from reality) (see ref. 13 for more precise asumptions). This proves that at the scale of the mean free path l, the flow remains laminar. So the effects of fully developed turbulence do not seem to break down the derivation of the slip boundary conditions.

Note that, according to the above statements, when ε is very small, the slip boundary conditions are close to the usual no-slip boundary conditions.

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